

Classical momentum diffusion in double- δ -kicked particles

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We investigate the classical chaotic diffusion of atoms subjected to *pairs* of closely spaced pulses (“kicks”) from standing waves of light (the 2δ -KP). Recent experimental studies with cold atoms implied an underlying classical diffusion of a type very different from the well-known paradigm of Hamiltonian chaos, the standard map. The kicks in each pair are separated by a small time interval $\epsilon \ll 1$, which together with the kick strength K , characterizes the transport. Phase space for the 2δ -KP is partitioned into momentum “cells” partially separated by momentum-trapping regions where diffusion is slow. We present here an analytical derivation of the classical diffusion for a 2δ -KP including all important correlations which were used to analyze the experimental data. We find an asymptotic ($t \rightarrow \infty$) regime of “hindered” diffusion: while for the standard map the diffusion rate, for $K \gg 1$, $D \sim K^2/2[1 - 2J_2(K) \dots]$ oscillates about the uncorrelated rate $D_0 = K^2/2$, we find analytically, that the 2δ -KP can equal, but never diffuses faster than, a random walk rate. We argue this is due to the destruction of the important classical “accelerator modes” of the standard map. We analyze the experimental regime $0.1 \leq K\epsilon \leq 1$, where quantum localization lengths $L \sim \hbar^{-0.75}$ are affected by fractal cell boundaries. We find an approximate asymptotic diffusion rate $D \propto K^3\epsilon$, in correspondence to a $D \propto K^3$ regime in the standard map associated with the “golden-ratio” cantori.

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I. INTRODUCTION

The “ δ -kicked particle” (δ -KP) is one of the best known examples of classical Hamiltonian chaos. A particle, or in an experiment usually a large ensemble of ultracold atoms, is periodically “kicked” by a series of very short laser pulses forming standing waves of light. The effective potential takes a sinusoidal form $V(x, t) = -K \cos x \sum_N \delta(t - NT)$. Here T is the kicking period, while K is the kick strength, related to the laser intensity. The classical dynamics for the δ -KP are given by the textbook example of chaos which is the “standard map” [1]. For large K , the dynamics is characterized by diffusion in momentum. To lowest order, this represents a random walk in momentum, hence $\langle p^2 \rangle \sim D_0 t$, where $D_0 \sim K^2/2$. The quantum counterpart of the δ -KP is the quantum kicked particle (QKP). It has also been extensively investigated in numerous theoretical (see e.g., [2–4]) and experimental studies in Austin [5], Auckland [6], Oxford [7], Lille [8], and Otago [9].

However, a recent experimental and theoretical study [10] of cold cesium atoms exposed to closely spaced *pairs* of pulses (the 2δ -kicked particle) showed chaotic classical diffusion quite different from all other previously studied δ -kicked systems. The two kicks in each pair are separated by a short-time interval $\epsilon \ll T$.

The cold cesium atoms are, of course, a realization of the quantum counterpart of the 2δ -KP; like the single-kick QKP the experiment exhibits the quantum chaos phenomenon of dynamical localization [2–4], whereby the quantum diffusion is arrested at a characteristic time scale, the “break-time,” $t^* \sim K^2/\hbar^2$. The momentum distribution of the atomic cloud is “frozen” for times $t > t^*$ with a momentum localization length $\langle p^2 \rangle \sim L^2$. By adjusting ϵ and hence the time scales of the diffusion correlations, relative to the break-time, it was found in [10] that the experiment probed distinct diffusive regimes characterized by different “families” of long-ranged

correlations. In the standard map, the main corrections to the uncorrelated random walk come from 2- and 3-kick correlations. For the 2δ -KP there were also families of terms correlating all kicks. Hence these additional 2δ -KP corrections were termed “global correlations.”

A particularly interesting feature of the 2δ -KP is the “cellular” structure of the classical phase space. This structure is analyzed in detail below, but the basic idea is illustrated in Fig. 1. An ensemble of particles, all initially with momentum $p = p_0$ diffuse chaotically, but it is seen that the diffusion is hindered at “trapping regions” i.e., regions with momenta $p \approx \pm(2m+1)\pi/\epsilon$ and $m = 0, 1, 2, \dots$. For small $K\epsilon$ trajectories spend most of their time stuck in the trapping regions. Once the particles have escaped, the time scale taken to diffuse freely over the remainder of the cell is negligible.

The experiments in [10] showed that for short and intermediate times (~ 100 kicks for the experimental parameters) the diffusion rates depended strongly on p_0 and t . Here we examine also for the first time, to the best of our knowledge, the asymptotic regime ($t \rightarrow \infty$) where the momentum spread of the atomic cloud is large compared to a single cell. At very long times momentum-dependent correlations decay to zero, leaving linear diffusion corrections leading to $\langle p^2 \rangle = D_\infty t$. Figure 2 provides a summary of the short time and the asymptotic momentum diffusion regimes in the 2δ -KP.

The regime $0.1 \leq K\epsilon \leq 1$ is of special interest because it corresponds roughly to the experimental parameters and because a new study of the *quantum* 2δ -KP [11,12] found a scaling behavior quite different from the usual QKP for this regime: while for the QKP the localization length $L \sim \hbar^{-1}$, for the 2δ -KP, $L \sim \hbar^{-0.75}$. To date this result is not fully explained. Here, we find $D_\infty \sim K^3\epsilon$ in this regime.

The structure of the paper is as follows: In Sec. II we review the classical dynamics of the standard map and the 2δ -KP. We explain the cellular structure of the 2δ -KP and we show it is natural to work in rescaled momenta $p_\epsilon = p\epsilon$.

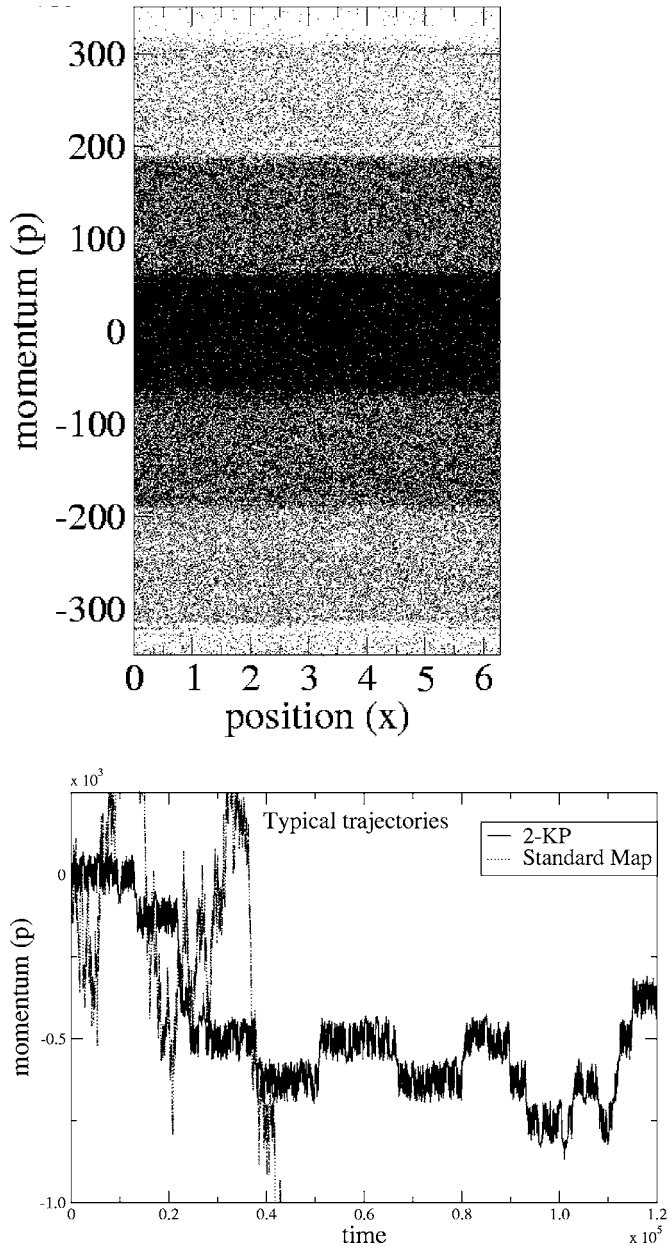


FIG. 1. (Top) Surface of section plot for the 2δ -KP, $K=7$, $\epsilon=0.05$, $\tau=T-\epsilon=1.95$, for atoms all with initial momentum $p_0=0$. The cellular structure is evident: momentum space is divided into regions of fast momentum diffusion separated by porous boundaries, i.e., narrow trapping regions where classical trajectories “stick” for relatively long periods. The trapping regions are at momenta $p \approx \pm(2m+1)\pi/\epsilon$ where $m=0,1,2,\dots$: (Bottom) A typical trajectory of the 2δ -KP compared with a standard map trajectory: the 2δ -KP trajectory spends considerable time trapped in a cell before escaping onto the next; the standard map trajectory looks like a simple random walk.

This removes effects dependent on the cell size $2\pi/\epsilon$ and also leads to an effective value of the kick strength $K_\epsilon=K\epsilon$. We obtain an approximate map for the trapping regions, from which we present a heuristic justification for the $D_\infty \sim K^3\epsilon$ behavior which is seen in the $K\epsilon \leq 1$ regime (where fractional power-law behavior was found for the quantum localization lengths $L \sim \hbar^{-0.75}$ [11,12]).

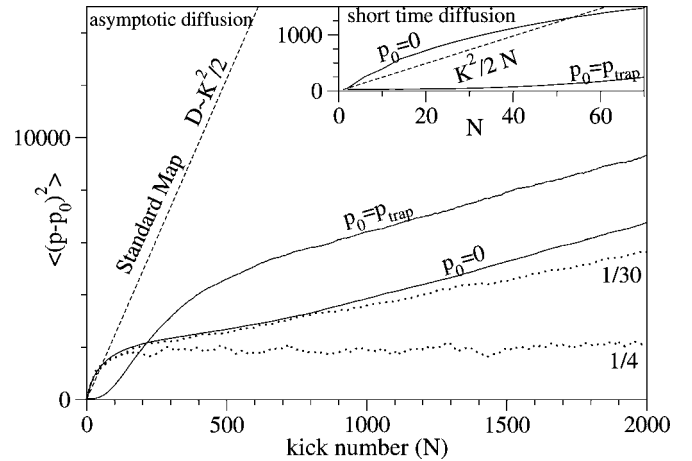


FIG. 2. Classical diffusion of the standard map vs the 2δ -KP. Here $K=7$ in both cases. For the standard map (broken lines) for $K \gg 1$, the same diffusion rate $\sim K^2/2$ characterizes all time scales and all starting conditions. For the 2δ -KP ($\epsilon=0.05$), the diffusion rate $D(p_0, t)$ at first depends on the initial momentum p_0 and time. The inset magnifies the $N < 50$ behavior. For atoms prepared near the cell boundaries, i.e., $p_0=p_{\text{trap}}=(2m+1)\pi/\epsilon$, where $m=0,1,2,\dots$, we find $D(p_0=p_{\text{trap}}, t)=0$ for the first 20 or so kicks, but then diffusion speeds up rapidly. For atoms prepared at the center of the cells, $p_0=2m\pi$, there is initially a rapid diffusion until the ensemble reaches the cell boundaries. We see that as $t \rightarrow \infty$ (in this case, for $N \geq 200$), the diffusion rate reaches an asymptotic value $D(p_0, t) \equiv D_\infty$ independent of p_0 . But $D_\infty < D_0$: the diffusion rate is always slower than the uncorrelated standard map. The quantum behavior is illustrated by the dotted lines. For effective $\hbar = \frac{1}{4}$, the quantum diffusion follows the classical behavior for only a few kicks before it is arrested by dynamical localization. For smaller $\hbar \approx \frac{1}{30}$ it follows it for considerably longer.

In Sec. III we explain how to derive the form of the long-range correlations which modify the diffusion rate of the 2δ -KP. While these results were used to analyze the experiments in [10], due to space constraints a full derivation could not be presented there. The emphasis is on time scales comparable to the quantum break-time t^* : since the diffusive behavior is “frozen-in” at this point, even transient (but long-lived) classical correlations become important for the quantum dynamics and hence are essential in understanding the experimental data. This section is the most technical however, and readers without a detailed interest in classical correlations can obtain the key points from Figs. 5–7 and their captions.

In Sec. IV we investigate the asymptotic linear diffusion regime ($t \rightarrow \infty$). We attempt to derive analytical forms for D_∞ . For large $K\epsilon$, one of the main results of the present work is the formula $D_\infty \approx K^2[1 - J_1^2(K\epsilon)/1 - J_0^2(K\epsilon)]$ which gives extremely accurate results compared to numerics (see Fig. 9). It is shown from this that the diffusion rate of the 2δ -KP can approach, but not exceed the uncorrelated rate (i.e., the value expected if the particles executed a random walk). We attribute this to the absence of accelerator modes [14] which have been observed experimentally for the usual QKP [15] but are not found in the 2δ -KP. For $K\epsilon \leq 1$, although we show that including higher-order families of long-ranged cor-

relations makes the analytical diffusion *tend* to the numerical result we are unable to quantitatively reproduce the $D_\infty \sim K^3 \epsilon$ behavior. Finally, in Sec. V, we conclude.

II. CLASSICAL DYNAMICS

A. Standard map

The classical standard map is obtained by integrating Hamilton's equations for motion in the δ -kick potential $V(x, t) = -K \cos x \sum_N \delta(t - NT)$. One obtains two equations which may be solved iteratively to evolve the system through each period T :

$$p_{N+1} = p_N + K \sin x_N; \quad x_{N+1} = x_N + p_{N+1}T. \quad (1)$$

A δ -kick is followed by a period of free evolution with constant momentum. With increasing kick-strength K the system makes a transition from integrable, regular dynamics to eventual full chaos. For $K \geq 1$, chaotic momentum diffusion is unbounded and all chaotic phase-space regions are connected. If we neglect all correlations between impulses, i.e. assuming $\langle \sin x_N \sin x_{N'} \rangle \approx 0$ for all kicks, the momentum of a trajectory in effect represents a 'random walk'. The corresponding energy of an ensemble of particles grows linearly with time, since $\langle p^2 \rangle = D_0 N \approx K^2 / 2N$. NB: from Sec. III B onwards, time is measured in kick-pairs $t = N/2$; in that case all diffusion rates are doubled; e.g., $\langle p^2 \rangle \sim D_0 t = K^2 t$; hence the uncorrelated diffusion rate in those units would be given by $D_0 = K^2$.

The overall diffusion in the chaotic regime is in general however not uncorrelated (provided K is not too large). In [13], the effect of short-range correlations between kicks was investigated theoretically. A more accurate form for the diffusion rate $D \approx K^2 [\frac{1}{2} - J_2(K) - J_1^2(K) \dots]$ was obtained, where $J_m(x)$ is a regular Bessel function of the first kind of order m and argument x . The second term is a 2-kick correction resulting from correlations $\langle \sin x_N \sin x_{N+2} \rangle$; the third term is a 3-kick correction resulting from $\langle \sin x_N \sin x_{N+3} \rangle$. The effects of these corrections on the energy absorbed by atoms in pulsed optical lattices have been experimentally observed [15]. Note that for the standard map, the correlations represent a simple change in the magnitude of D ; the energy increase is still linear in time. In [16] it was further shown that in an asymmetric potential, the 2-kick correlations yield a local correction to the diffusion, i.e., D depends on both time and the relative initial momentum, p_0 , between the atoms and the standing wave of light. This produces a type of chaotic Hamiltonian ratchet.

B. The 2 δ -KP

The classical map for the 2 δ -KP is a straightforward extension of the standard map:

$$p_{N+1} = p_N + K \sin x_N; \quad p_{N+2} = p_{N+1} + K \sin x_{N+1}$$

$$x_{N+1} = x_N + p_{N+1} \epsilon; \quad x_{N+2} = x_{N+1} + p_{N+2} \tau, \quad (2)$$

where ϵ is a very short time interval between two kicks in a pair and τ is a much longer time interval between the pairs. It

is easily seen from the map that atoms for which $p_0 \epsilon = (2m + 1)\pi$ and $m = 0, 1, 2, \dots$, experience an impulse $K \sin x_0$ followed by an impulse $\approx K \sin(x_0 + \pi)$, which in effect cancels the first. The regime $p_0 \approx (2m + 1)\pi / \epsilon$ corresponds to the "momentum-trapping" regions. Conversely in the case $p_0 \epsilon = 2m\pi$, a series of near-identical kicks produces initially rapid energy growth.

Some of the characteristics of the diffusion can be analyzed by the properties of the classical map in the trapping regions. Starting from the map (2) with $N=0$ we rescale all variables $p^\epsilon = p \epsilon$ and $K_\epsilon = K \epsilon$, $\tau_\epsilon = \tau / \epsilon \gg 1$ to obtain

$$p_1^\epsilon = p_0^\epsilon + K_\epsilon \sin x_0; \quad p_2^\epsilon = p_1^\epsilon + K_\epsilon \sin x_1$$

$$x_1 = x_0 + p_1^\epsilon; \quad x_2 = x_1 + p_2^\epsilon \epsilon. \quad (3)$$

We take $p_0^\epsilon = p_R^\epsilon + \delta p^\epsilon$, where p_R^ϵ is the trapping momentum, i.e., $p_R^\epsilon = (2m + 1)\pi$ and we choose $m=0$ for the first trapping region. Inserting the above into 3 we have

$$p_2^\epsilon = p_0^\epsilon + K_\epsilon \sin x_0 + K_\epsilon \sin(x_0 + \pi + \delta p^\epsilon + K_\epsilon \sin x_0)$$

$$= p_0^\epsilon + K_\epsilon \sin x_0 + K_\epsilon [\sin(x_0 + \pi) \cos(\delta p^\epsilon + K_\epsilon \sin x_0)$$

$$+ \cos(x_0 + \pi) \sin(\delta p^\epsilon + K_\epsilon \sin x_0)]. \quad (4)$$

Assuming small angle identities throughout, $\cos(f(\epsilon)) \approx 1$ and $\sin(f(\epsilon)) \approx f(\epsilon)$,

$$p_2^\epsilon \approx p_0^\epsilon - K_\epsilon \cos x_0 [K_\epsilon \sin x_0 + \delta p^\epsilon]$$

$$= p_0^\epsilon - \frac{K_\epsilon^2}{2} \sin 2x_0 - K_\epsilon \delta p^\epsilon \cos x_0. \quad (5)$$

So at the center of the trapping region ($\delta p^\epsilon = 0$) the double-kick map is equivalent to an effective $\sin 2x$ single-kick standard map:

$$p_2^{\epsilon R} \approx p_0^\epsilon - K_R \sin 2x_0 \quad (6)$$

where $K_R = K_\epsilon^2 / 2$. Further away from the exact trapping momentum where $\delta p^\epsilon \gg K_\epsilon / 2$ we have a cosinusoidal map

$$p_2^{\epsilon \delta p} \approx p_0^\epsilon - K_{\delta p} \cos x_0, \quad (7)$$

where $K_{\delta p} = K_\epsilon \delta p^\epsilon$.

From these arguments we clearly see that the natural stochasticity parameter of the 2 δ -KP in rescaled momenta $p^\epsilon = p \epsilon$ is $K_\epsilon = K \epsilon$. It is also important to estimate the range of $0 < |\delta p^\epsilon| < \delta p_{max}^\epsilon$: the trapping regions have a small but finite width determined by whether there is significant cancellation between consecutive kicks. Requiring $K_\epsilon \sin x_0 + K_\epsilon \sin(x_0 + \pi + \delta p^\epsilon) \approx 0$ we estimate $\delta p_{max}^\epsilon \approx \pi / 6$, in other words over about a sixth of the width of each momentum cell, classical trajectories experience significant trapping in the momentum diffusion.

In Figs. 3 and 4 we can see the change from a $\sin 2x$ map at $p^\epsilon \approx p_R^\epsilon$, where we observe two sets of stable islands within the range $0 < x < 2\pi$, to a cosinusoidal map further out where the position of islands is shifted by a phase of $\pi/2$. Figure 3 compares the detailed structure at $p^\epsilon \approx p_R^\epsilon$ with a standard map phase space for which $K_{SM} = K^2 \epsilon / 2$. Figure 4 compares

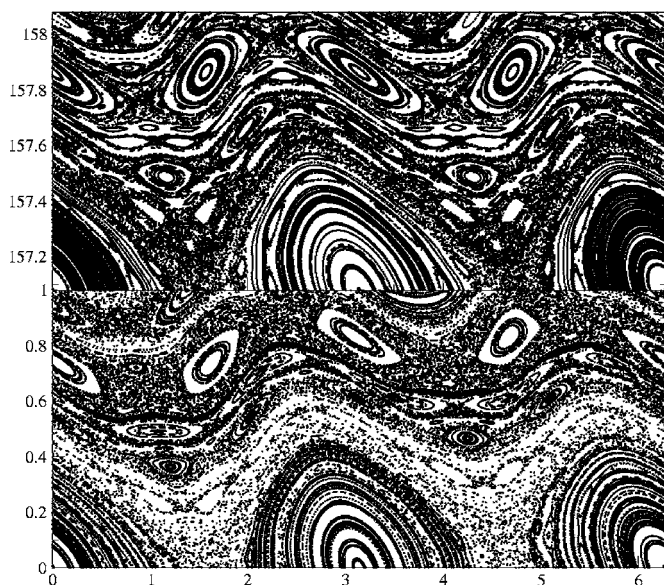


FIG. 3. Correspondence between the local phase space at the resonant momentum p_R of a trapping region ($K=4.8$, $\epsilon=0.02$, $\tau=1.98$) in the double-kicked system (top) and a standard map (bottom) with a kick $p_2=p_0-K_R \sin 2x_0$, where $K_R=K^2\epsilon/2=0.24$ is the kick strength of the standard map and the period is $T=2$. Note that unscaled momenta are used here.

an enlarged section of a “side resonance” further away from p_R^ϵ , with a standard map with $K_{SM}=K\epsilon\delta p$. In both cases the correspondence is clearly visible.

We note that in the standard map there is a transition to unbounded diffusion for a critical kick strength $K=K_{crit}\approx 1$ where the last invariant curve breaks, leaving a fractal cantorus structure. This curve corresponds to momenta $p\approx 2\pi mR$, where R is the golden ratio. In [19] it was shown that for $K_{crit}<K\leq 4.5$ the diffusion rate is $D\approx 0.3(K-K_{crit})^\eta$. The value $\eta\approx 3$ was obtained in [19] from the scal-

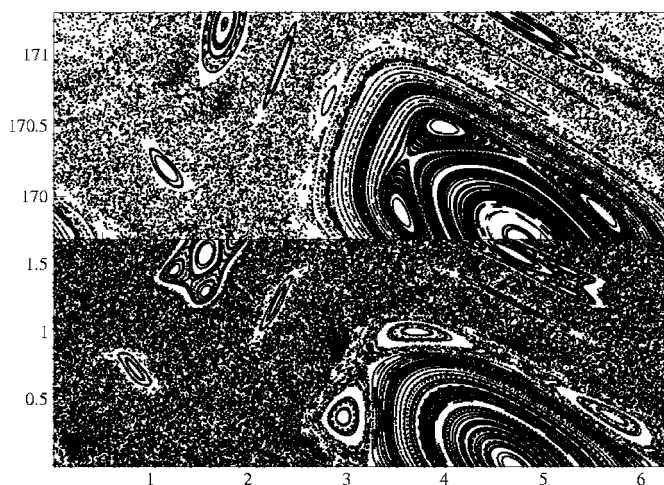


FIG. 4. Correspondence between the outer phase space near the resonant momentum p_R of a trapping region ($K=4.0$, $\epsilon=0.02$, $\tau=1.98$) in a double-kicked system (top) and a $\pi/2$ phase-shifted cosinusoidal standard map (bottom) $p_2=p_0-K_{\delta p} \cos x_0$, where $K_{\delta p}=K\epsilon\delta p=0.75$ is the kick strength of the standard map and the period is $T=2$. Note that unscaled momenta are used here.

ing properties of the classical map near the golden-ratio cantorus.

From the above, one can hazard a rough order-of-magnitude justification for a cubic dependence of the asymptotic diffusion rate $D_\infty\approx K^3\epsilon$ for $0.1\leq K_\epsilon\leq 1$ found numerically (see Sec. IV). Note that in scaled momenta, $D_\infty\sim K_\epsilon^3$. We note the correspondence between the standard map and the 2δ -KP with $K\rightarrow K_\epsilon$; for small K_ϵ , $\delta p^\epsilon>0$, the form of the map in 7 is dominant, since trajectories spend most of their time in the trapping regions. In this case, the scaled critical kick strength $K_\epsilon^{crit}\approx\epsilon$. Taking a representative value of $\langle\delta p^\epsilon\rangle\approx\delta p_{max}^\epsilon/2=\pi/12\approx\frac{1}{4}$, the effective kick strength in 7 is $\approx K_\epsilon/4$. We thus have a diffusion rate in the trapping regions in terms of rescaled momenta $\langle p_\epsilon^2\rangle\sim 0.3(K/4-K_\epsilon^{crit})^3t$. Then, in unscaled momenta, we would have $D\sim K^3\epsilon$ provided that K is reasonably large compared with ϵ . While this is by no means a rigorous argument it may provide some indication of the mechanism underpinning the K^3 dependence. We note the 0.75 exponent in the $L\propto\hbar^{-0.75}$ scaling is also an exponent associated with scaling properties of the region around the golden-ratio cantori. We propose tentatively that this supports the proposal in [11,12] that the properties of the 2δ -KP are intimately connected with the properties of the golden-ratio cantorus.

III. MOMENTUM DIFFUSION

A. Standard map

The classical diffusion corrections for the standard map were first obtained by Rechester and White [13] and we follow their notation closely. From the map (1), the momenta and positions of a trajectory evolve by a sequence of impulses: $p_N=p_0+S_{N-1}$ and $x_N=x_{N-1}+p_0+S_{N-1}$, where $S_l=\sum_{j=0}^l K \sin x_j$, the initial momentum of an atom is p_0 and the period T is taken to be unity. If we consider an ensemble of particles with an initial probability distribution in position and momentum $G(x_0, p_0, t=0)$, at a later time (measured in number of kicks $t=N$) the distribution is given by

$$G(x_t, p_t, t) = \sum_{n_t=-\infty}^{+\infty} \cdots \sum_{n_1=-\infty}^{+\infty} \int_0^{2\pi} dx_0 dp_0 G(x_0, p_0, 0) \\ \times \int_0^{2\pi} dx_t \cdots \int_0^{2\pi} dx_1 \delta(p_t - p_0 - S_{t-1}) \\ \times \delta(x_t - x_{t-1} - p_0 - S_{t-1} + 2\pi n_t) \cdots \delta(x_1 - x_0 \\ - p_0 - S_0 + 2\pi n_1). \quad (8)$$

The sums over n_1, \dots, n_t appear because of the periodicity of phase space in x_0, \dots, x_t . The momentum diffusion rate D is given by

$$D = \lim_{t\rightarrow\infty} \frac{1}{t} \langle (p_t - p_0)^2 \rangle_t = \frac{1}{t} \int_0^{2\pi} dx_t \int_{-\infty}^{+\infty} dp_t G(x_t, p_t, t) (p_t - p_0)^2. \quad (9)$$

By taking the initial distribution as $G(x_0, p_0, 0)=1/2\pi\delta(p-p_0)$ (i.e., a uniform spatial distribution with all particles at

initial nonzero momentum p_0) and using the Poisson sum formula giving the Fourier transform of a δ spectrum, $\sum_n \delta(y+2\pi n) = 1/2\pi \sum_m e^{imy}$, we can rewrite (9) as

$$D = \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_t=-\infty}^{\infty} \prod_{i=0}^t \int_0^{2\pi} \frac{dx_i}{2\pi} (S_{t-1})^2 \times \exp\left(\sum_{j=1}^t im_j(x_j - x_{j-1} - p_0 - S_{j-1})\right). \quad (10)$$

To lowest order one can set all m_j coefficients to zero, thus eliminating all exponentials. By using the previous form of S and integrating over the sine products it is easily seen that the random walk $D_0 = K^2/2$ is recovered. Higher-order corrections to the diffusion rate are obtained by setting certain m_j coefficients to a nonzero value; for the most dominant corrections $|m_j| = 1, 2$. The integrals are solved using the relation $\exp\{\pm iK \sin x\} = \sum_{n=-\infty}^{+\infty} J_n(K) \exp\{\pm inx\}$ and for corrections to be nonzero, all arguments of exponentials must vanish for 2π -periodic integration. This requires pairing exponentials with the relevant Bessel functions and sine products included in S .

The main corrections to the standard map are the 2-kick and 3-kick correlations found in [13] and account in large measure for the experimental oscillations seen in [15]. The 2-kick correlation is obtained from setting $m_j = \pm 1$ and $m_{j-1} = \mp 1$ and the lowest-order 3-kick correlation from $m_j = \pm 1$ and $m_{j-2} = \mp 1$. From the $\sin x_j \sin x_{j-2}$ and $\sin x_j \sin x_{j-3}$ terms we obtain $C_2 = -K^2 J_2(K)$ and $C_3 = -K^2 J_1^2(K)$, respectively, as previously. There is also a higher-order 3-kick correlation, $C'_3 = +K^2 J_3^2(K)$, found in [13] derived from $m_j = \pm 1$, $m_{j-1} = \mp 2$, $m_{j-2} = \pm 1$, and a 4-kick correlation, $C_4 = +K^2 J_2^2(K)$, cited in [20] derived from $m_j = \pm 1$, $m_{j-1} = m_{j-2} = \mp 1$, $m_{j-3} = \pm 1$. This leads to a total correction to the diffusion of

$$D = K^2 \left[\frac{1}{2} - J_2(K) - J_1^2(K) + J_3^2(K) + J_2^2(K) \right]. \quad (11)$$

These terms represent the correlations between two given impulses only, two, three, or four kicks apart. In the next section we shall see that in the double-kicked system there are entire families of terms representing correlations between a given impulse $\sin x_j$ and every other impulse. Such a global correlation family does, in fact, exist for the standard map, as the above method of derivation can be extended to any order of k -kick correlations between any two given kicks [17] ($\sin x_j \sin x_{j-k}$). For the lowest order we have $m_j = \pm 1$ and $m_{j-k+1} = \mp 1$ and hence $C_k = -K^2 J_1^2(K) J_0^{k-3}(K)$ ($k \geq 3$). Clearly corrections become smaller with increasing k . The 2-kick correlation is a special case of this global collection of terms. Note the linear time dependence of all individual k -kick correlations (i.e., $\langle p^2 \rangle \propto C_k t$). The total correction to the momentum diffusion rate due to all the above k -kick terms is $\sum_{k=3}^{\infty} C_k = -K^2 [J_1^2(K)] / [1 - J_0(K)]$. Similar higher-order global corrections can be found in the standard map of the general form $\pm K^2 [\prod_p J_p^m(K)] / \{\prod [1 - J_0(K)]^n\}$ for some m, n, p . However, in practice, higher-order corrections beyond the basic terms in [20] can generally be neglected for all $K \geq 5$ since they do not alter the diffusion rates significantly. For smaller

K there can be significant differences; however phase space becomes increasingly regular as K decreases and for $K \leq 2$ a diffusive approach is not justified. In the double-kick system the higher-order (long-range) correlations originate only in the thin trapping layers so a diffusive analysis is effective even in a regime where long-range correlations are important.

It should be noted that none of these corrections depends on the momenta of the atoms in the ensemble; they simply alter the magnitude of the overall linear rate of energy absorption for a given K . In any δ -kicked system where the kicking periods are all equal and there are no other asymmetries, effects associated with momentum-dependent diffusion corrections are negligible. This is due to such corrections including oscillations $\cos T p_0$ on a comparable scale to the natural width of the initial momentum distribution $\Delta p_0 \sim 2\pi$ of the atomic ensemble. Hence these corrections average to zero. The 1-kick correlation $\sin x_j \sin x_{j-1}$ between consecutive kicks for which a single $m_j = \pm 1$ only, is such a correction and is hence absent in the standard map, but will be seen to contribute significantly to the double-kicked system, where a much shorter kicking period is introduced.

B. The 2 δ -KP

For the 2 δ -KP the notation of the standard map diffusion equation can be changed slightly to include two closely spaced kicks for each time step denoted by (1) and (2); thus the evolution of momentum is now in terms of pairs of kicks. Throughout this section we work in unscaled momenta $\langle p^2 \rangle = Dt$ (in rescaled momenta $\langle p_\epsilon^2 \rangle = D\epsilon^2 t$). S takes on the form $S_t^{(2)} = \sum_{m=1}^t \sum_{r=1}^2 K \sin x_m^{(r)}$ [17] and we indicate explicitly the two time intervals τ and ϵ , defined as previously. Again, D_0 is obtained by setting all m_j coefficients to zero, but it should be noted that as there are now $2t$ variables, the form of D_0 changes to K^2 and hence $\langle (p-p_0)^2 \rangle = D_0 t = K^2 t$ (or in scaled momenta $K_\epsilon^2 t$). Obviously this does not change the underlying physics; the new formula is only due to a redefinition of time in terms of the number of pairs of kicks, $t = N/2$, for even N . Physical time is $t(\tau + \epsilon)$.

$$D = \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{m_t^{(2)}=-\infty}^{\infty} \sum_{m_t^{(1)}=-\infty}^{\infty} \cdots \sum_{m_1^{(2)}=-\infty}^{\infty} \sum_{m_1^{(1)}=-\infty}^{\infty} \prod_{i=1}^t \int_0^{2\pi} \frac{dx_i^{(2)}}{2\pi} \times \int_0^{2\pi} \frac{dx_i^{(1)}}{2\pi} (S_t^{(2)})^2 \exp\left(\sum_{j=1}^t \{im_j^{(2)}[x_j^{(2)} - x_j^{(1)} - \epsilon(p_0 + S_j^{(1)})] + im_j^{(1)}[x_j^{(1)} - x_{j-1}^{(2)} - \tau(p_0 + S_{j-1}^{(2)})]\}\right). \quad (12)$$

C. Momentum-dependent diffusion of the 2 δ -KP

The introduction of the short time scale ϵ results in a significant 1-kick contribution to the diffusion, unlike in any previously studied δ -kicked system. For $m_j^{(2)} = \pm 1$ the correlation involves $\cos p_0 \epsilon$ and Bessel functions of argument $K\epsilon = K_\epsilon$, while for $m_j^{(1)} = \pm 1$ the correlation involves $\cos p_0 \tau$ and Bessel functions of argument $K\tau$. The latter case gives

negligible contributions as τ is the large-time interval between pairs of kicks, resulting in fast oscillations with p_0 . We can effectively set all $m_j^{(1)}=0$ for all momentum-dependent correlations—a valid approximation provided $\tau \gg \epsilon$. The $m_j^{(2)}$ coefficients will be simply referred to as m_j for the remainder of the paper.

For the case of $\cos p_0 \epsilon$, the $\sin x_j^{(2)} \sin x_j^{(1)}$ term results in a kick-to-kick correction

$$C_1 t = K^2 \cos p_0 \epsilon [J_0(K_\epsilon) - J_2(K_\epsilon)] \sum_{j=1}^t J_0^{2j-2}(K_\epsilon), \quad (13)$$

where $t=N/2$ (N even) i.e., time is measured in pairs of kicks as for all correlations that follow. Kicks at longer times have weaker correlations than those at short times, however, since $J_0(K_\epsilon) \rightarrow 1$ as $\epsilon \rightarrow 0$, the time-dependent summation decays slowly with time and the lifetime of the overall kick-to-kick correlation may far exceed the active running time of an experiment. Note however that the correlation does decay to zero eventually (as for all other momentum-dependent correlations), i.e., $D(p_0, t \rightarrow \infty) = 0$. Hence we do not take the infinite time limit in 12; instead we calculate the quantity $C_1 t$. The geometric sum in 13 saturates to $1/(1-J_0^2)$ after a time $\sim 10/(K_\epsilon)^2$ and hence

$$\lim_{t \rightarrow \infty} C_1 t = K^2 \cos p_0 \epsilon \frac{J_0(K_\epsilon) - J_2(K_\epsilon)}{1 - J_0^2(K_\epsilon)}. \quad (14)$$

For short times $C_1 t$ grows linearly with time, since for small K_ϵ we have $J_0(K_\epsilon) \approx 1 \gg J_2(K_\epsilon)$. We can approximate the correlation to $K^2 t \cos p_0 \epsilon$ and hence the average energy of the double-kicked particle also grows linearly in this regime.

Figure 5 is the same as in [10] showing a numerical simulation of the energy absorption of an ensemble of 100 000 classical particles at $K=7$, $\epsilon=0.05$, $\tau=T-\epsilon=1.95$, as a function of their initial momenta p_0 at various times (measured in pairs of kicks). The numerics are superposed with combinations of the time-dependent correlations $C_j t$ presented in this paper. In Fig. 5(a) the basic cosine behavior of C_1 is clearly visible: atoms with initial momenta $p_0 \approx 2m\pi/\epsilon = 0, 125.66, \dots$, experience the largest energy absorption, while those prepared at $p_0 \approx (2m+1)\pi/\epsilon = 62.83, 188.50, \dots$, are “trapped” near this initial momentum and experience almost no energy absorption. When one looks at Figs. 5(c) and 5(d) something unexpected occurs. The maxima of Fig. 5(a) slowly turn into near minima, while energy absorption for atoms near the “trapping” regions increases continuously. A complete reversal of the initial situation eventually occurs at longer times, $\langle (p-p_0)^2 \rangle \propto -\cos p_0 \epsilon$.

The reasons for this lie in considering a whole family of global correlations similar to the case of the standard map. The lowest-order global correction originates from $m_j = \pm 1$, just as for C_1 , but this time we look at $\sin x_j^{(2)} \sin x_i^{(r)}$ terms, where $r=1, 2$ and $i < j$ but otherwise arbitrary. In this way we include all k -kick correlations for $k \geq 2$, i.e., correlations between the second impulse in a given pair, $\sin x_j^{(2)}$, and impulses in all other pairs.

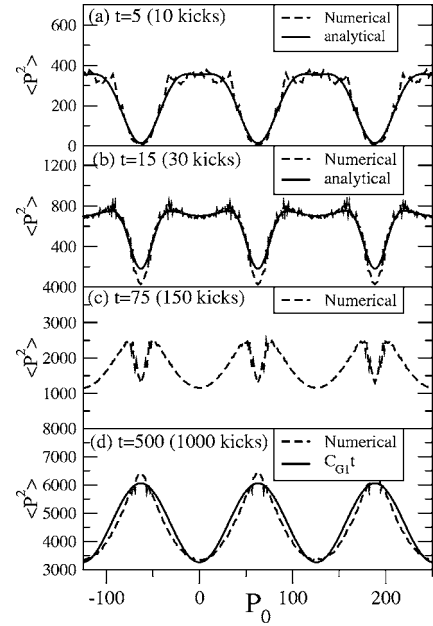


FIG. 5. Agreement between analytical diffusion corrections and numerical double-kick simulations of 100 000 classical particles at $K=7$, $\epsilon=0.05$, $\tau=1.95$, for various times measured in pairs of kicks, $t=N/2$ where N is the (even) number of individual kicks. Physical time is $t(\tau+\epsilon)$ where ϵ is the short time interval between kicks in a pair and τ is the long time interval between pairs. Energy absorption is plotted as a function of initial momentum of the particles. The corrections included in the analytical curve in (a) are $C_1 t$, $C_{G1} t$ and the lowest-order $C_{G2}^{(1)} t$ term labeled IIa in Table I. In (b) all correlations given in Table I are included. Agreement is excellent at short times, but higher-order terms are needed at later times. One observes a sign reversal of the cosine envelope and gradual disappearance of the inverted Poisson peaks at initial minima. Particles initially prepared in momentum-trapping regions eventually absorb the most energy. At very long times the lowest order global correlation C_{G1} dominates almost completely.

$$C_{G1} t = \sum_{k=2}^{2t-1} C_k t = -2K^2 \cos p_0 \epsilon J_1^2(K_\epsilon) \sum_{j=1}^t (2j-2) J_0^{2j-3}(K_\epsilon). \quad (15)$$

The corrected energy is now $\langle (p-p_0)^2 \rangle \approx K^2 t + C_1 t + C_{G1} t$. Every individual correlation between two kicks in C_{G1} is small compared to the nearest neighbor correlation C_1 , as $J_1^2(K_\epsilon) \ll J_0(K_\epsilon)$. Importantly however, at a given time the correlations do not get weaker with increasing k but are equal in size for any non-nearest neighbor correlation. All correlations, however, do become weaker with increasing time and eventually saturate. Summing over t reference impulses and $2j-2$ paired correlations for each, results in a total correction to the diffusion which after a given time becomes dominant relative to C_1 . Note the difference in sign between the 1-kick correlation and all others. The summation in (15) can be seen to be the derivative of that in (13) and hence

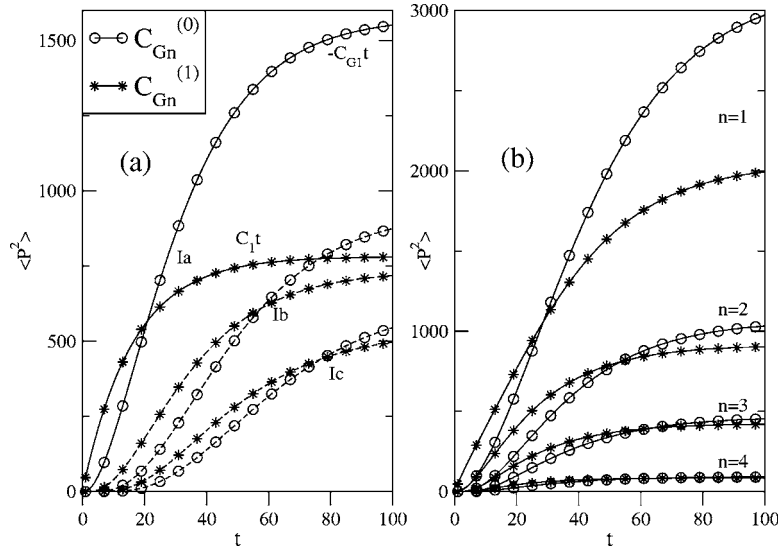


FIG. 6. Behavior of global correlations $C_{Gn}t$ with number of pairs of kicks. Absolute values $|C|t$ of maximum energy absorption are shown ($\cos np_0\epsilon=1$). (a) shows the comparison between individual correlations at order $n=1$, including nearest neighbor C_1 , lowest-order global C_{G1} , and higher-order Poisson correlations. (b) shows the comparison between different cosine orders n . Trapping terms $C_{Gn}^{(1)}$ are shown with stars, absorption enhancing terms $C_{Gn}^{(0)}$ with circles. The labels correspond directly to those in Table I showing explicit values at $t=15$. Terms are denoted by the cosine order n resulting from the total sum of nonzero m_j coefficients, and the order in J_1 resulting from the number of nonzero coefficients. Curves in (b) are made up of the various correlations in Table I for a given order n . Note the linear and quadratic rise, respectively, for C_1 and C_{G1} at early times. Absorption enhancing terms always overtake their momentum trapping partners at equal order after sufficient time, but the crossing point shifts to later times as both the order n and $O(J_1)$ increases. Overall importance of terms also decreases for higher orders.

$$C_{G1}t = -4K^2 \cos p_0\epsilon J_1^2(K_\epsilon) \times \frac{J_0(K_\epsilon) - tJ_0^{2t-1}(K_\epsilon) + (t-1)J_0^{2t+1}(K_\epsilon)}{[1 - J_0^2(K_\epsilon)]^2}. \quad (16)$$

For small K_ϵ it can be shown by assuming $J_0(K_\epsilon) \approx 1 - K_\epsilon$ and expanding the above to second-order binomial that $C_{G1}t$ initially increases quadratically at short times:

$$\begin{aligned} 1 - K_\epsilon - t(1 - K_\epsilon)^{2t-1} + (t-1)(1 - K_\epsilon)^{2t+1} \\ \approx 1 - K_\epsilon - t[1 - (2t-1)K_\epsilon + (2t^2 - 3t + 1)K_\epsilon^2] \\ + (t-1)[1 - (2t+1)K_\epsilon + (2t^2 + t)K_\epsilon^2] \\ = 2K_\epsilon^2(t^2 - t). \end{aligned} \quad (17)$$

For the parameters in Fig. 5, $C_{G1}t$ saturates to a value approximately twice that of C_1t as $t \rightarrow \infty$.

$$\lim_{t \rightarrow \infty} C_{G1}t = -4K^2 \cos p_0\epsilon \frac{J_0(K_\epsilon)J_1^2(K_\epsilon)}{[1 - J_0^2(K_\epsilon)]^2}. \quad (18)$$

The behavior of the 1-kick and all global correlations with time is illustrated in Fig. 6 (including higher-order Poisson correlations presented later). The absolute value ($|C|t$) of the maximum energy absorption (where $\cos p_0\epsilon=1$) with an increasing number of kick pairs is shown. Note that all momentum-dependent correlations have a nonlinear time dependence and saturate after sufficient time. This means that the momentum-dependent diffusion rate approaches zero at long times, where all particles absorb the same amount of energy irrespective of p_0 . At short times the effect of C_{G1} is

negligible compared to C_1 , but at later times the global correlations dominate the diffusive process, explaining the sign reversal in Fig. 5(d). The importance of the global correlations persists for values of K_ϵ for which the corresponding phase space is completely chaotic, a phenomenon not previously observed.

Figure 5(b) corresponds to a regime where C_1 and C_{G1} are of similar importance (near the crossing point in Fig. 6). Here and in Fig. 5(c) another feature mentioned earlier becomes evident. The initial troughs of the cosine in Fig. 5(a) turn into narrow downward peaks, superimposed onto the cosine envelope. The origins of these peaks are global correlation terms of higher cosine orders $\cos np_0\epsilon$, which yield near δ functions at the relevant momenta through $\sum_n (-1)^n \cos np_0\epsilon = \sum_m \delta(p_0\epsilon - (2m+1)\pi)$. Such Poisson terms arise when more than one m_j coefficient in 12 is set to ± 1 , and $\sum_j m_j = n$, i.e., the sum of the coefficients defines the cosine order n . Terms with $|m_j| > 1$ include Bessel functions of increasingly higher orders; since for $\epsilon \rightarrow 0$, $J_n(K_\epsilon) \rightarrow 0$ more rapidly for increasing $n > 0$, such terms can be neglected here.

Derivations of higher-order global correlations are tedious, yet reasonably straightforward. There are two distinct families, one following the pattern of C_1 and the other following C_{G1} , each causing different diffusive behavior. In both cases the terms of interest involve $\sin x_j^{(2)} \sin x_k^{(r)}$ where $r=1,2$ and $k < j$. In the former case $m_k^{(2)} = \pm 1$ and correlations involve a $J_0 - J_2$ factor as for C_1 . In the latter case $m_k^{(2)} = 0$ and correlations involve a $2J_1$ factor as for C_{G1} . The behavior depends on whether the correlation is with a kick in a pair associated with a corresponding zero or nonzero m_k

coefficient. We choose to denote the global Poisson terms as $C_{Gn}^{(0)}$ and $C_{Gn}^{(1)}$ depending on the family they belong to, where n is the cosine order. C_{G1} is the lowest-order global correlation family belonging to $C_{G1}^{(0)}$. All Poisson terms have the following general forms for some m :

$$C_{Gn}^{(1)}t = (-1)^{n-1}K^2 \cos np_0 \epsilon J_1(nK_\epsilon) \left(\prod_m J_1^2(mK_\epsilon) \right) \times \left(\sum_m \frac{J_0(mK_\epsilon) - J_2(mK_\epsilon)}{J_1(mK_\epsilon)} \right) \sum_j \sum_{\alpha_m=0}^{f(j)} \left(\prod_m J_0^{2\alpha_m}(mK_\epsilon) \right), \quad (19)$$

$$C_{Gn}^{(0)}t = (-1)^n 2K^2 \cos np_0 \epsilon J_1(nK_\epsilon) \times \left(\prod_m J_1^2(mK_\epsilon) \right) \sum_j \sum_{\alpha_m=0}^{f(j)} \left(\sum_m 2\alpha_m \frac{J_1(mK_\epsilon)}{J_0(mK_\epsilon)} \right) \times \left(\prod_m J_0^{2\alpha_m}(mK_\epsilon) \right), \quad (20)$$

where $f(j)$ depends on the particular term: some examples of Poisson terms are given in the Appendix. Each additional $m_k = \pm 1$ coefficient adds two multiplicative J_1 factors of possibly different arguments to both Poisson families, since Bessel function arguments can now be any integer multiple of K_ϵ , depending on the exact combination of m_k coefficients. This continuously decreases the significance of the correlations, since $J_1(x) \rightarrow 0$ as $x \rightarrow 0$. Hence the dominant corrections are those of lowest order in J_1 (and with few nonzero m_k coefficients). Note that for Poisson terms other than C_{G1} various k -kick correlations are of similar but not necessarily equal strength at a given time t .

For $n=1$ there are higher-order Poisson terms (in J_1) contributing only to the cosine envelope in Fig. 5. These are shown in Fig. 6(a) in comparison to the lowest-order corrections $C_1 t$ and $C_{G1} t$. Again absolute values of maximum energy absorption are shown. As $O(J_1)$ increases, terms become less significant. For all terms, $C_{G1}^{(0)}$ correlations eventually become more dominant than $C_{G1}^{(1)}$ correlations, but the crossing point shifts to later times with increasing $O(J_1)$. The difference in saturation energies between the two types of correlations also becomes smaller.

Figure 6(b) shows the comparison between the combined $n=1$ correlation families and higher cosine orders. The behavior with increasing n is as before; $C_{Gn}^{(0)}$ correlations always become more dominant than their partners of equal order, after sufficient time. $C_{Gn}^{(1)}$ terms increase as $\sim t^n$, while $C_{Gn}^{(0)}$ terms increase as $\sim t^{n+1}$.

Table I shows the maximum energy absorption values of the dominant Poisson terms at $t=15$ (30 kicks) corresponding to Fig. 5(b). The most significant diffusion corrections are those for small n and $O(J_1)$ and importance of terms rapidly decreases with higher orders. Terms of $O(J_1^{10})$ are about 50 times smaller than the leading corrections.

The momentum diffusion corrections derived here now enable us to explain the behavior seen in Fig. 5, including the inverted peaks. The agreement between the numerical data and the analytical diffusion calculations is generally good. At

TABLE I. Diffusion correlations shown in Fig. 6.

Term	m_j pattern	$O(J_1)$	Value ($t=15$)
Ia (C_1, C_{G1})	± 1	0,2	+472, -355
Ib ($C_{G1}^{(1)}, C_{G1}^{(0)}$)	$\pm 1, \pm 1, \mp 1$	4,6	+100, -33
Ic ($C_{G1}^{(1)}, C_{G1}^{(0)}$)	$\pm 1, \pm 1, \pm 1, \mp 1, \mp 1$	8,10	+14, -4
IIa ($C_{G2}^{(1)}, C_{G2}^{(0)}$)	$\pm 1, \pm 1$	2,4	-227, +113
IIb ($C_{G2}^{(1)}, C_{G2}^{(0)}$)	$\pm 1, \pm 1, \pm 1, \mp 1$	6,8	-59, +24
IIIa ($C_{G3}^{(1)}, C_{G3}^{(0)}$)	$\pm 1, \pm 1, \pm 1$	4,6	+82, -39
IIIb ($C_{G3}^{(1)}, C_{G3}^{(0)}$)	$\pm 1, \pm 1, \pm 1, \pm 1, \mp 1$	8,10	+29, -12
IVa ($C_{G4}^{(1)}, C_{G4}^{(0)}$)	$\pm 1, \pm 1, \pm 1, \pm 1$	6,8	-29, +14

very short times [Fig. 5(a)] essentially only C_1 , C_{G1} , and $C_{G2}^{(1)}$ (IIa) contribute to the diffusion and the agreement is excellent. In Fig. 5(b) all of the correlations in Table I have been included in the analytical curve and good agreement is obtained: the sign of the cosine envelope starts to change and the inverted peaks slowly vanish at later times. From 19 and 20 one notes that at the same order n , the two types of correlations are of different sign and so oppose each other, clearly seen in the case of $n=1$. For $n > 1$ it is found that while $C_{Gn}^{(1)}$ correlations contribute to increasing the size of the downward peaks in all cases and thus favor momentum trapping, $C_{Gn}^{(0)}$ correlations result in enhanced energy absorption for initially trapped atoms. These latter correlations cause the release of atoms from trapping regions in momentum space.

In Fig. 5(c) the analytical curve has been omitted as a reasonable agreement cannot be achieved using only the correlations in Table I. Higher-order terms are needed at intermediate times, however at very long times the diffusion is dominated by C_{G1} , the most important correlation, as can also be seen in Fig. 6(a). After saturation, the higher-order $n > 1$ Poisson correlations result in superimposed upward peaks at the maxima of the cosine envelope giving the latter a pointed appearance. The newly found global correlation families lead to a situation where at long times, when diffusion no longer depends on initial momentum, those atoms that started in a trapping region have actually gained more energy than those that started in an enhanced absorption region in momentum space.

So far this paper has only treated the diffusion problem classically, but a real experiment would obviously be carried out in the quantum regime. In [10,12,18] it was shown that the effects presented here can readily be observed in a quantum experiment, and indeed a range of other interesting features were discovered.

As mentioned in the Introduction to this paper, energy absorption does not continue indefinitely in the quantum case. The energy saturates to a near-constant value after a characteristic quantum break time $t^* \sim D/\hbar^2$ [2], where \hbar is a scaled Planck constant. The smaller the \hbar , the longer the system follows the classical predictions (see Fig. 2).

Figure 7 shows the energy absorption of a cloud of cold atoms for $K=3.3$ and $\hbar=1$, measured in the 2δ -KP experiment in [10]. Values of ϵ vary from 0.045 to 0.160 such that $K_\epsilon=0.1485, 0.3102, \text{ and } 0.528$. The resemblance to Fig. 5 is evident and is due to these different values of K_ϵ which con-

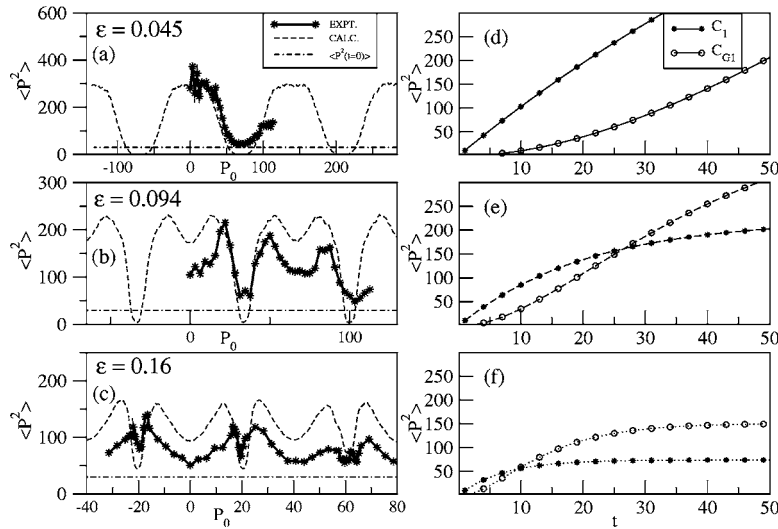


FIG. 7. Experimental realization of the 2δ -KP with a cloud of cold cesium atoms, pulsed periodically by pairs of laser kicks. The graphs show the energy absorption of the ensemble for $K=3.3$, $\epsilon=0.045, 0.094, 0.160$, and $\hbar=1$. Measurements are taken after the quantum break time, $t^* \sim 30$, when the energies have saturated and momentum diffusion has been terminated. Varying K_ϵ means that the time scale t_G on which the global diffusion corrections become dominant varies in relation to t^* . In (a) $t^* < t_G$ and the system is arrested in a regime where C_1 dominates, corresponding to 5(a). Atoms prepared in the trapping regions remain trapped and absorb no energy. In (b) $t^* \approx t_G$ and the effects of the global corrections C_{Gn} become visible: inverted Poisson peaks appear at trapping momenta and a partial sign reversal of the overall cosine envelope is seen. In (c) finally, $t^* > t_G$ and the system follows classical energy trajectories long enough for the global corrections to become dominant. Trapped atoms start to absorb energy and the diffusion in other parts of the phase space approaches a $(-\cos p_0\epsilon)$ relationship. (d)–(f) show the behavior of C_1 (stars) and C_{G1} (circles) with time for the values of K_ϵ in (a)–(c) ($\tau=2-\epsilon$). We can consider t_G to be near the crossing point of the two correlations shown. This clearly shifts to earlier times as K_ϵ increases.

control the relative importance of diffusion correlations. It should be noted that in experiments, ϵ and τ are usually dimensionless scaled quantities, such that $\epsilon = \Delta/T$, where Δ is the physical time between kicks in a pair and T is the physical period of the double-kick system. The measurements are all taken well after the break time $t^* \approx D_0 \sim 30$, when the energies have saturated and no further evolution takes place. The experimental behavior then depends on which diffusion corrections are dominant at the break time. As K_ϵ decreases the time needed for the global diffusion corrections to dominate over C_1 increases, hence in Fig. 7(a) the system is arrested at a time where C_1 is still hugely dominant. In Figs. 7(b) and 7(c), however, K_ϵ is larger and the global correlations C_{Gn} become more important when the break time is reached. In these cases the Poisson peaks and sign reversals for the cosine envelope are clearly seen. Figures 7(d)–7(f) show C_1 and C_{G1} for the parameters in (a)–(c), and the crossing point, indicating the point at which the global correlations become dominant, clearly shifts from $t_G > t^*$ to $t_G < t^*$ at higher K_ϵ . Saturation values of correlations also decrease with increasing time between the double kicks, which results in the limit of the standard map where $\epsilon = \tau \approx 2$ and there is no momentum-dependent diffusion at all. Note that at very high values of K , and hence K_ϵ , correlations become less significant due to large Bessel arguments [$J_n(x) \rightarrow 0$ as $x \rightarrow \infty$].

From the momentum distribution of the atomic ensemble one can also observe the trapping regions and the effects of the global diffusion corrections. After the quantum break time, the distribution of the usual δ -KP localizes exponentially in the momentum basis, causing the characteristic

triangular shape on a logarithmic plot. For the 2δ -KP the basic exponential shape has a “staircase” superposed onto it (see [11,12]).

IV. ASYMPTOTIC DIFFUSION: $t \rightarrow \infty$

Classically, the momentum-dependent diffusion in the 2δ -KP does not continue indefinitely but is only a transient effect at short and intermediate times. At very long times in the asymptotic regime, momentum-dependent correlations saturate and the diffusion rate is the same for all starting conditions p_0 . In this regime diffusion is controlled by linear corrections independent of the initial momentum, as for the standard map, which decrease in magnitude with increasing separation between kicks but remain constant in time.

Such corrections are obtained from Eq. (12) in the case where the total sum of m_j coefficients is zero; hence $n=0$ and $\cos np_0\epsilon=1$. It is found that all these corrections are of opposite sign to the random walk $D_0=K^2$ and hence lower the overall rate of energy absorption for all values of K_ϵ . This is due to the absence of terms for which $|m_j^{(2)}|=2$ or $m_j^{(1)} \neq 0$. Although for momentum-independent correlations, terms can also depend on the long time scale τ between kick pairs, i.e., $m_j^{(1)} = \pm 1$, similar to the standard map case ($\tau \approx T$), such terms are far less significant than the dominant ϵ -dependent corrections. Hence k -kick correlations for even k , such as the important 2-kick correction ($\propto J_2$), are effectively absent in the 2δ -KP. The general form of correlations dependent on ϵ only is

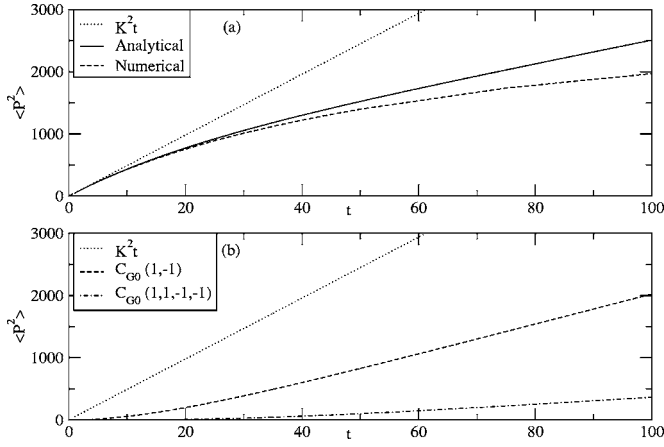


FIG. 8. Effect of momentum-independent corrections to the overall rate of energy absorption by the 2δ -KP. (a) shows the significant decrease in energy absorption from the basic random walk, $D_0=K^2$. The analytical curve is composed of the corrections in (b) and shows reasonable agreement with the numerical data. Parameters are as in Fig. 5. Absolute values of the correlations are plotted in (b), so the analytical curve in (a) is $K^2 t - C_{G0} t(\pm 1, \mp 1) - C_{G0} t(\pm 1, \mp 1, \mp 1, \mp 1)$.

$$C_{G0} t = -K^2 \left(\prod_m J_1^2(mK_\epsilon) \right) \sum_j \sum_{\alpha_m=0}^{f(j)} \left(\prod_m J_0^{\alpha_m}(mK_\epsilon) \right). \quad (21)$$

Note the difference in sign in comparison to D_0 and absence of Bessel functions of order $p > 1$. Note further that although we denote these correlations by Ct , the actual diffusion rate $D(t \rightarrow \infty)$ is nonzero and can be found by taking the infinite time limit of 21. From the lowest-order correlation for which $m_j = \pm 1, m_k = \mp 1$ ($k < j$) we obtain a global family similar to the case of the standard map, including the usual lowest-order 3-kick correlation

$$C_{G0} t(\pm 1, \mp 1) = -K^2 J_1^2(K_\epsilon) \sum_{j=2}^t \sum_{\alpha=0}^{j-2} J_0^{2\alpha}(K_\epsilon) \quad (22)$$

and hence, for $K_\epsilon \gg 1$,

$$D \approx K^2 \left(1 - \frac{J_1^2(K_\epsilon)}{1 - J_0^2(K_\epsilon)} \right). \quad (23)$$

Note the similarity to the lowest-order global family C_k presented for the standard map earlier. The minor difference of a J_0^2 factor stems from the fact that this family includes only the case of odd k . Higher-order corrections for the 2δ -KP have the asymptotic form $D = -K^2 \prod_m [J_1^2(mK_\epsilon)] / [1 - J_0^2(mK_\epsilon)]^n$ for some m, n .

Figure 8(a) shows the effect of the two lowest-order global correlation families on the energy absorption of the kicked particles for the same parameters as in Fig. 5 ($K_\epsilon = 0.35$). The change from the K^2 random walk is seen to be very significant. It should be noted that at early times $C_{G0} t$ behaves nonlinearly; however, this is only due to the inclusion of entire global families of correlations between all kicks. As time increases, more correlations are included in

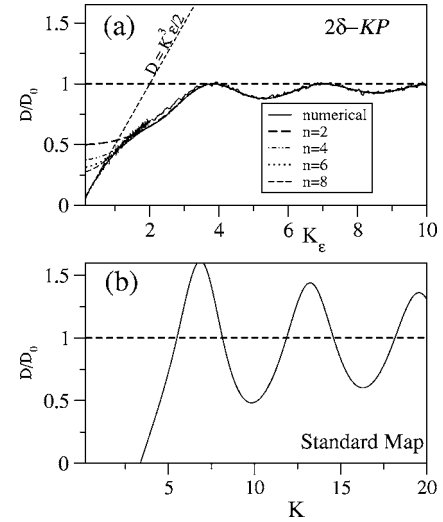


FIG. 9. (a) The diffusion rate for the 2δ -KP (shown as a function of stochasticity parameter $K_\epsilon = K\epsilon$) never exceeds the “random walk” rate $D_0 = K^2$; the uncorrelated rate (for time counted in kick pairs), is $D_0 = K^2$. The figure shows that for large K_ϵ , there is an excellent agreement between Eq. (23) and numerics; in the regime of cold atom experiments, $K_\epsilon \lesssim 1$, dominated by fractal phase-space structures, agreement with the numerical result is improved by including higher-order correlations [$O(J_1^n)$] but remains nonquantitative. Note that the numerical result $D \approx K^2 \epsilon / 2$ is represented by a straight (dashed) line since $D/D_0 = K_\epsilon / 2$. (b) Illustrates the “textbook” standard map diffusion rate $D \approx K^2 / 2 [1 - 2J_2(K) \dots]$ obtained by Rechester and White [13]. The rate oscillates about the random walk value and the maxima occur where classical accelerator modes are important.

the formula, however, each individual k -kick correction still has a linear time dependence, as for all correlations independent of momentum. At long times the energy increase becomes linear again, since corrections become weaker with increasing k .

Figure 8(b) shows absolute values of the two corrections included in the analytical curve in 8(a) in comparison to $D_0 = K^2$. It is seen that the lowest-order term given in 22 makes up the vast majority of the diffusion correction. The second term (given in the Appendix) is derived from a sequence of $m_j = \pm 1, \pm 1, \mp 1, \mp 1$ and is hence $O(J_1^6)$. The agreement between the analytical and numerical data in 8(a) is reasonable and can be improved by including higher-order terms.

Figure 9 shows a comparison of the ratio D/D_0 between the standard map and the 2δ -KP. Figure 9(b) shows the well-known diffusion behavior for the case of the standard map or δ -KP where $D_0 = K^2/2$ and D is given by 11 [this curve was first presented in [13] but without the $C_4 = K^2 J_2^2(K)$ term]. D/D_0 vs K is shown and the behavior is oscillatory around the usual random walk with some regimes of enhanced and some of hindered diffusion. At very large K the diffusion approaches the random walk value asymptotically. As was mentioned in Sec. II, higher-order global corrections do not alter the shape of the curve in 9(b) appreciably. For $K \lesssim 5$ a linear relationship is found where $D \propto K^3$, however as K is decreased the system becomes increasingly regular and the

diffusion corrections no longer describe the behavior accurately.

Figure 9(a) shows D/D_0 vs K_ϵ for the 2δ -KP, where $D_0=K^2$ (in unscaled momenta) and a series of analytical curves are shown, including an increasing number of global correction families, in comparison to the numerically obtained diffusion behavior. The analytical curves are denoted by the maximum order of $J_1(mK_\epsilon)$ included. The behavior for large K_ϵ is found to be different from the case of the standard map [seen in 9(b)]: for the 2δ -KP the diffusion is always “hindered” compared to the random walk; i.e., $D < D_0$ for all K_ϵ , while for the standard map, there are maxima in the diffusion where $D > D_0$. For the latter, the maxima correspond to values of the stochasticity parameter $K \approx 2\pi M$, where M is an integer. These coincide with the appearance of small transporting islands (i.e., trajectories in these islands gain $\approx 2\pi$ in momentum for each kick), the so-called accelerator modes, see e.g., [14]; the surrounding chaotic phase space generates Levy flights. This regime of anomalous diffusion was observed in the experiments in [15]. We have found no evidence whatsoever of any accelerator modes for $K \approx 2\pi M$; we cannot exclude a small contribution from other higher-order accelerator modes, but the small maxima of the D/D_0 in the 2δ -KP correspond to zeros of $J_1(K_\epsilon)$ and the diffusion is bounded from above by the random walk rate. Asymptotically, $D \rightarrow D_0$ as $K_\epsilon \rightarrow \infty$ as all correlations decay away completely.

For small K_ϵ we again have a near-linear section where $D \propto K^3 \epsilon$ (or $D \propto K_\epsilon^3$ in scaled momenta p^ϵ) as mentioned in Sec. II. The agreement between analytical and numerical results is excellent for $K_\epsilon \gtrsim 2$, where even the lowest-order [$O(J_1^2)$] correlations are sufficient to reproduce the diffusion curve. As $K \rightarrow 0$ agreement can be improved by including successively higher-order corrections.

V. CONCLUSIONS

We have analyzed the chaotic classical diffusion of the double δ -KP. Although a straightforward extension of the standard map, this system exhibits a rich variety of new features.

One motivation of the present work was to further understand the behavior observed in experiments [10] and to present a more detailed derivation of the formulas in [10]. The experiment probed regimes where the classical diffusion is highly nonlinear in time and depends on initial conditions, but which were nevertheless sufficiently generic to be analyzed in terms of corrections to a diffusive process. This is in contrast to the standard map, where long-range correlations are a feature of the near-integrable regime where most trajectories are stable so an analysis based on diffusive processes is not useful. Although the nonlinear regime of the 2δ -KP is, classically, transient, it is sufficiently long-lived (~ 100 – 200 kicks) so that it is the *only* regime sampled by the quantum dynamics of the experiment in [10].

Quantum studies of the 2δ -KP in the regime $0.1 \leq K_\epsilon \leq 1$ were originally undertaken because they coincide roughly with the experimental values. However, the theoretical studies then revealed novel quantum behavior in the en-

ergy level statistics and the fractional scalings $L \sim \hbar^{-0.75}$ of the quantum localization lengths [11,12]. It was suggested that the *global* quantum properties of the 2δ -KP are, in this regime, intimately connected with the *local* scalings around the neighborhood of the golden-ratio cantori, as transport from cell to cell is limited by the former. The K^3 dependence of the diffusion rate found here lends further credence to this suggestion, as a cubic dependence is a feature of diffusion near golden-ratio cantori [20]. Unfortunately, as for the standard map, the cubic behavior cannot be derived analytically. At best, it was seen that adding successively higher orders of diffusive corrections makes the analysis tend towards the numerics.

One of the main results here is contained in Fig. 9, and Eq. (23): in the regime $K_\epsilon \gg 1$, where effects of cantori and trapping are negligible. Here, the diffusion of the 2δ -KP follows a simple analytical expression but still differs strikingly from the standard map, in that it never exceeds the random walk rate (though it can equal it). The 2δ -KP has no 2-kick correction [the $J_2(K)$ term which in the standard map roughly “tracks” the accelerator modes]. We have found no numerical evidence for the accelerator mode behavior. Hence, although there is much evidence for nonlinear diffusion (of finite though prolonged duration) in the 2δ -KP, the major source of anomalous diffusion seen in the chaotic standard map is, in fact, absent; in this sense, the 2δ -KP diffusion is closer to the random walk (as shown in Fig. 9) once trapping by cantori becomes unimportant.

Finally, it is worth noting the potential applications of the 2δ -KP, in particular, as a velocity-selective atom filter in, for example, devices such as an atom chip. Narrow trapping regions could be used to select atoms with $p_0 \approx \pi/\epsilon$, while others would accelerate through the system nearly unperturbed. This could also be used to create very pure Bose-Einstein condensates, if located in a trapping region. A much stronger momentum-dependent effect is seen in the double-kick system than in the previously studied perturbed-period system [16]. Other applications in atomic manipulation may also be possible given further investigations.

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APPENDIX: GLOBAL CORRELATION FAMILIES

Exact forms of two of the momentum-dependent global correlation families presented in this paper are given here as examples, as well as one further example of a higher-order C_{G0} correlation term. Note that the $m_j^{(2)}$ patterns given here and also shown in Table I correspond to the sum of correlations with such nonzero coefficients, including all possible permutations. The sum of the coefficients defines the cosine order n and permutations for which the partial sum, when coefficients are added from the highest to lowest j , is zero, lead to zero-valued correlations.

$$m_j = \pm 1, \pm 1, \mp 1 \text{ (Term Ib in Table I)}$$

$$C_{G1}^{(1)}t = -K^2 \cos p_0 \epsilon J_1^3(K_\epsilon) J_1^2(2K_\epsilon) \left(3 \frac{J_0(K_\epsilon) - J_2(K_\epsilon)}{J_1(K_\epsilon)} + 2 \frac{J_0(2K_\epsilon) - J_2(2K_\epsilon)}{J_1(2K_\epsilon)} \right) \times \sum_{j=3}^t \sum_{\alpha=0}^{j-3} J_0^{2\alpha}(K_\epsilon) J_0^{2(j-3-\alpha)}(2K_\epsilon), \quad (\text{A1})$$

$m_j = \pm 1, \pm 1$ (Term IIa in Table I)

$$C_{G2}^{(0)}t = 2K^2 \cos 2p_0 \epsilon J_1(2K_\epsilon) J_1^2(K_\epsilon) \sum_{j=2}^t \sum_{\alpha=0}^{j-2} \left(2\alpha \frac{J_1(K_\epsilon)}{J_0(K_\epsilon)} + 2(j-2-\alpha) \frac{J_1(2K_\epsilon)}{J_0(2K_\epsilon)} \right) J_0^{2\alpha}(K_\epsilon) J_0^{2(j-2-\alpha)}(2K_\epsilon), \quad (\text{A2})$$

$m_j = \pm 1, \pm 1, \mp 1, \mp 1$ [second higher-order correlation term in Fig. 8(b) and included in Fig. 9(a)]

$$C_{G0}t(\pm 1, \pm 1, \mp 1, \mp 1) = -K^2 [J_1^4(K_\epsilon) J_1^2(2K_\epsilon)] \times \sum_{j=4}^t \sum_{\alpha_1=0}^{j-4} \sum_{\alpha_2=0}^{j-4-\alpha_1} \sum_{\alpha_3=0}^{j-4-(\alpha_1+\alpha_2)} J_0^{2\alpha_1}(K_\epsilon) J_0^{2\alpha_2}(2K_\epsilon) J_0^{2\alpha_3}(K_\epsilon). \quad (\text{A3})$$

Other Poisson and higher-order global correlation terms have similar forms and are derived in the same way as the above from 12. Note that the time behavior ($C_{Gn}^{(1)}$ terms increase as $\sim t^n$, while $C_{Gn}^{(0)}$ terms increase as $\sim t^{n+1}$) for small K_ϵ and t can be shown for all terms by evaluating the sequence of geometric sums and expanding powers binomially to the appropriate order. Saturation values and asymptotic behavior are also straightforward to evaluate by taking the limit as $t \rightarrow \infty$.

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